Factorial Treatment Structure: Part I
Experimental units

- homogeneous
- inhomogeneous

CRD

- one-way ANOVA
  - fixed effects, global test, contrasts, ...
  - random effects, variance components, ...

Block Designs

- one block
- two (more)

RCB

- block size
  - large
  - small

- RCB with factorial treatment structure,
  - random effects, mixed effects models, nested factor structure, ...
  - split-plot, split-split plot designs, different models on whole- and subplots, ...

- (B)IBD

- Latin Squares

- Youden Squares

Similar to Lawson (2015)
Factorial Treatment Structure

- So far (in CRDs), the treatments had no “structure”.

- So called **factorial treatment structure** exists if the $g$ treatments are the **combination** of the **levels of two or more factors**.

- In the case that we see all the possible combinations of the levels of the two factors, we call the factors **crossed**.

- Examples
  - Biomass of crop: different **fertilizers** and different **crop varieties**.
  - Battery life: Different **temperature levels** and different **plate material** (Montgomery, 1991, Example 7-3.1).
  - ...

...
Example (Linder, A. und W. Berchtold, 1982)

- Response: Needleweight of 20 three-week old pine seedlings [in 1/100 g].
- Two factors:
  - $A$: “origin” with levels \{Taglieda, Pfyn, Rheinau\}
  - $B$: “exposure to light” with levels \{short, long, permanent\}
- We denote by $y_{ijk}$ the $k$th response of the treatment formed by the $i$th level of factor $A$ and the $j$th level of factor $B$. 

\[
\begin{align*}
\text{Factor } A & \text{ on level } i \\
& \text{(here: origin)} \\
\text{Factor } B & \text{ on level } j \\
& \text{(here: exposure)} \\
\text{Replicate number} & \quad (\text{here: } k = 1, 2) \\
\end{align*}
\]
# Two Factor Design: Generic Data Table

<table>
<thead>
<tr>
<th></th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$y_{111}$</td>
<td>$y_{121}$</td>
<td>$y_{131}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y_{112}$</td>
<td>$y_{122}$</td>
<td>$y_{132}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y_{113}$</td>
<td>$y_{123}$</td>
<td>$y_{133}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y_{114}$</td>
<td>$y_{124}$</td>
<td>$y_{134}$</td>
<td></td>
</tr>
<tr>
<td>$A_2$</td>
<td>$y_{211}$</td>
<td>$y_{221}$</td>
<td>$y_{231}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y_{212}$</td>
<td>$y_{222}$</td>
<td>$y_{232}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y_{213}$</td>
<td>$y_{223}$</td>
<td>$y_{233}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y_{214}$</td>
<td>$y_{224}$</td>
<td>$y_{234}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
## Data Table of Our Example

<table>
<thead>
<tr>
<th></th>
<th>Short</th>
<th>Long</th>
<th>Permanent</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Taglieda</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>42</td>
<td>62</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>38</td>
<td>55</td>
</tr>
<tr>
<td><strong>Pfyn</strong></td>
<td>45</td>
<td>62</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>42</td>
<td>58</td>
<td>75</td>
</tr>
<tr>
<td><strong>Rheinau</strong></td>
<td>50</td>
<td>52</td>
<td>88</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>62</td>
<td>95</td>
</tr>
</tbody>
</table>
Visualization

- As for one-way ANOVA situation
  - for all treatment combinations
  - factor-wise summaries (“marginal summaries”)

- More useful: **Interaction plot** (see R-code)
Factorial Treatment Structure

- The **structure of the treatment** influences the **analysis** of the data.

- **Setup:**
  - Factor $A$ with $a$ levels
  - Factor $B$ with $b$ levels
  - $n$ replicates for **every** combination
  - Total of $N = a \cdot b \cdot n$ observations

We could analyze this with the usual **cell means model** (ignoring the special treatment structure).

Typically, we have research questions about **both** factors and their possible **interaction** (interplay).
Factorial Treatment Structure

- Examples:
  - “Is effect of light exposure location specific?”
    (→ interaction between light exposure and location)
  - “What is the effect of light exposure averaged over all locations?”
    (→ main effect of light exposure)
  - “What is the effect of location averaged over all exposure levels?”
    (→ main effect of location)

- We could use the cell means (one-way ANOVA) model and try to answer these questions with appropriate contrasts (→ complicated).

- Easier: Use a model that incorporates the factorial structure of the treatments.
Factorial Model

The two-way ANOVA model with interaction is

\[ Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk} \]

where

- \( \alpha_i \) is the main effect of factor A at level \( i \).
- \( \beta_j \) is the main effect of factor B at level \( j \).
- \( (\alpha\beta)_{ij} \) is the interaction effect between A and B for level combination \( i, j \) (not the product \( \alpha_i \beta_j \)!)
- \( \epsilon_{ijk} \) are i.i.d. \( N(0, \sigma^2) \) errors.
- Typically, sum-to-zero constraints are being used, i.e.
  - \( \Sigma_{i=1}^{a} \alpha_i = 0, \Sigma_{j=1}^{b} \beta_j = 0. \) \( \rightarrow \) \( a - 1 \) and \( b - 1 \) degrees of freedom
  - \( \Sigma_{i=1}^{a}(\alpha\beta)_{ij} = 0, \Sigma_{j=1}^{b}(\alpha\beta)_{ij} = 0. \) \( \rightarrow (a - 1) \cdot (b - 1) \) degrees of freedom

because two factors involved
Interpretation of Main Effects

- **Main effects** are nothing else than the **average effect** when moving from row to row (column to column).

- **Interaction effect** is the difference to the main effects model, i.e. it measures how far the treatment means differ from the main effects model.

- If there is **no interaction**, the effects are **additive**.

- In our example it would mean: “No matter what location we are considering, the effect of light exposure is always the same.”
Visualization of Model

Factor $A$, Factor $B$ with two levels each

$$E[Y_{ijk}] = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$$
Visualization of Model

If no interaction is present, lines will be parallel.

**Interaction plot** is nothing else than empirical version of this plot.

\[
E[Y_{ijk}] = \mu + \alpha_i + \beta_j
\]
# Two-Way ANOVA Step By Step

## Data

<table>
<thead>
<tr>
<th>Location</th>
<th>Short</th>
<th>Long</th>
<th>Permanent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taglieda</td>
<td>25</td>
<td>42</td>
<td>62</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>38</td>
<td>55</td>
</tr>
<tr>
<td>Pfyn</td>
<td>45</td>
<td>62</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>42</td>
<td>58</td>
<td>75</td>
</tr>
<tr>
<td>Rheinau</td>
<td>50</td>
<td>52</td>
<td>88</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>62</td>
<td>95</td>
</tr>
</tbody>
</table>

## Parameters

- **μ**
- **α**
- **β**
- **ε**

## ANOVA Table

<table>
<thead>
<tr>
<th>Location</th>
<th>Short</th>
<th>Long</th>
<th>Permanent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taglieda</td>
<td>(\beta_1)</td>
<td>(\beta_2)</td>
<td>(\beta_3)</td>
</tr>
<tr>
<td></td>
<td>(\beta_1)</td>
<td>(\beta_2)</td>
<td>(\beta_3)</td>
</tr>
<tr>
<td>Pfyn</td>
<td>(\beta_1)</td>
<td>(\beta_2)</td>
<td>(\beta_3)</td>
</tr>
<tr>
<td></td>
<td>(\beta_1)</td>
<td>(\beta_2)</td>
<td>(\beta_3)</td>
</tr>
<tr>
<td>Rheinau</td>
<td>(\beta_1)</td>
<td>(\beta_2)</td>
<td>(\beta_3)</td>
</tr>
<tr>
<td></td>
<td>(\beta_1)</td>
<td>(\beta_2)</td>
<td>(\beta_3)</td>
</tr>
</tbody>
</table>

### Two-Way ANOVA Formulas

\[
\begin{align*}
\text{SS}_{\text{Total}} &= \sum (\text{observation} - \bar{\text{total}})^2 \\
\text{SS}_{\text{Between}} &= \sum n_k (\bar{y}_k - \bar{\text{total}})^2 \\
\text{SS}_{\text{Within}} &= \text{SS}_{\text{Total}} - \text{SS}_{\text{Between}} \\
\end{align*}
\]
Parameter Estimates

- Estimates for the balanced case (and the sum-to-zero constraints) are

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>$\hat{\mu} = \bar{y}_{..}$</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>$\hat{\alpha}<em>i = \bar{y}</em>{i..} - \bar{y}_{..}$</td>
</tr>
<tr>
<td>$\beta_j$</td>
<td>$\hat{\beta}<em>j = \bar{y}</em>{.j} - \bar{y}_{..}$</td>
</tr>
<tr>
<td>$(\alpha\beta)_{ij}$</td>
<td>$(\hat{\alpha}\hat{\beta})<em>{ij} = \bar{y}</em>{ij} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j$</td>
</tr>
</tbody>
</table>

- This means: we estimate the main effects as if the other factors wouldn’t be there (see next slides).
Main Effect of Location

- Original data-set

<table>
<thead>
<tr>
<th>Location</th>
<th>Short</th>
<th>Long</th>
<th>Permanent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taglieda</td>
<td>25</td>
<td>42</td>
<td>62</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>38</td>
<td>55</td>
</tr>
<tr>
<td>Pfyn</td>
<td>45</td>
<td>62</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>42</td>
<td>58</td>
<td>75</td>
</tr>
<tr>
<td>Rheinau</td>
<td>50</td>
<td>52</td>
<td>88</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>62</td>
<td>95</td>
</tr>
</tbody>
</table>

- Ignore factor “exposure”, estimate effect as in one-way ANOVA model

\[ \hat{\alpha}_i \]
Main Effect of Light Exposure

- Original data-set

<table>
<thead>
<tr>
<th></th>
<th>Short</th>
<th>Long</th>
<th>Permanent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taglieda</td>
<td>25</td>
<td>42</td>
<td>62</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>38</td>
<td>55</td>
</tr>
<tr>
<td>Pfyn</td>
<td>45</td>
<td>62</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>42</td>
<td>58</td>
<td>75</td>
</tr>
<tr>
<td>Rheinau</td>
<td>50</td>
<td>52</td>
<td>88</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>62</td>
<td>95</td>
</tr>
</tbody>
</table>

- Ignore factor “location”, estimate effect as in one-way ANOVA model

\[ \hat{\beta}_j \]
Sum of Squares

- Again, total sum of squares can be **partitioned** into different sources, i.e. $SS_T = SS_A + SS_B + SS_{AB} + SS_E$.

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Sum of squares (SS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$a - 1$</td>
<td>$\sum_{i=1}^{a} b \cdot n \cdot \hat{\alpha}_i^2$</td>
</tr>
<tr>
<td>$B$</td>
<td>$b - 1$</td>
<td>$\sum_{j=1}^{b} a \cdot n \cdot \hat{\beta}_j^2$</td>
</tr>
<tr>
<td>$AB$</td>
<td>$(a - 1) \cdot (b - 1)$</td>
<td>$\sum_{i=1}^{a} \sum_{j=1}^{b} n \cdot (\hat{\alpha}_i \cdot \hat{\beta}_j)^2$</td>
</tr>
<tr>
<td>Error</td>
<td>$(n - 1) \cdot ab$</td>
<td>$\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \bar{y}_{ij})^2$</td>
</tr>
<tr>
<td>Total</td>
<td>$abn - 1$</td>
<td>$\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \bar{y}_{..})^2$</td>
</tr>
</tbody>
</table>

- Squared effect
- # observations with that effect
- Fitted value
- Product of df's of involved factors
- # observations – 1 – sum(df above)
- Degrees of freedom of total
### ANOVA Table

- As before, we can construct an **ANOVA table**

<table>
<thead>
<tr>
<th>Source</th>
<th>$df$</th>
<th>$SS$</th>
<th>$MS$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$a - 1$</td>
<td>$SS_A$</td>
<td>$\frac{SS_A}{a - 1}$</td>
<td>$\frac{MS_A}{MS_E}$</td>
</tr>
<tr>
<td>$B$</td>
<td>$b - 1$</td>
<td>$SS_B$</td>
<td>$\frac{SS_B}{b - 1}$</td>
<td>$\frac{MS_B}{MS_E}$</td>
</tr>
<tr>
<td>$AB$</td>
<td>$(a - 1) \cdot (b - 1)$</td>
<td>$SS_{AB}$</td>
<td>$\frac{SS_{AB}}{(a - 1)(b - 1)}$</td>
<td>$\frac{MS_{AB}}{MS_E}$</td>
</tr>
<tr>
<td>Error</td>
<td>$ab \cdot (n - 1)$</td>
<td>$SS_E$</td>
<td>$\frac{SS_E}{(n - 1)ab}$</td>
<td></td>
</tr>
</tbody>
</table>

- Under the corresponding (global) null-hypothesis, the $F$-ratio is again $F$-distributed with degrees of freedom defined through the numerator and the denominator, respectively.
$F$-Tests: Overview

- **Interaction $AB$**
  - $H_0$: $(\alpha \beta)_{ij} = 0$ for all $i, j$
  - $H_A$: At least one $(\alpha \beta)_{ij} \neq 0$
  - Under $H_0$: \( \frac{MS_{AB}}{MS_E} \sim F_{(a-1)(b-1), ab(n-1)} \)

- **Main effect $A$**
  - $H_0$: $\alpha_i = 0$ for all $i$
  - $H_A$: At least one $\alpha_i \neq 0$
  - Under $H_0$: \( \frac{MS_A}{MS_E} \sim F_{(a-1), ab(n-1)} \)

- **Main effect $B$**
  - $H_0$: $\beta_j = 0$ for all $j$
  - $H_A$: At least one $\beta_j \neq 0$
  - Under $H_0$: \( \frac{MS_B}{MS_E} \sim F_{(b-1), ab(n-1)} \)
Analyzing the ANOVA Table

- Typically, the \( F \)-tests are analyzed from **bottom to top** (in the ANOVA table).
- Here, this means we **start with the \( F \)-test of the interaction.**
- If we reject:
  - Conclude that we need an interaction in our model, i.e. the effect of \( A \) depends on the level of \( B \).
  - **Don’t continue** testing the main-effects (**principle of hierarchy**)
  - Perform individual analysis for every level of factor \( A \) (or \( B \)), error of **full model** can be re-used (see later).
  - Have a look at **interaction plot** to get a better understanding of what is going on.
  - Interaction might be based on a **single** cell.
  - Transformation might help in getting rid of the interaction.

- If we can’t reject, continue testing the main effects.
Two-Way ANOVA model in R

- Define model with interaction in formula interface

```r
> fit <- aov(y ~ location * exposure, data = data)
> summary(fit)

             Df Sum Sq Mean Sq F value Pr(>F)
location      2 2053.1 1026.4 69.981 3.28e-06 ***
exposure      2  407.4 203.71 138.890 1.72e-07 ***
location:exposure 4  183.3  45.78  3.117  0.0722 .
Residuals     9  132.2  14.70
---
Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

- We can also use the following equivalent notation

```r
> fit <- aov(y ~ location + exposure + location:exposure, data = data) ## equivalent version
```
More than Two Factors

- Model can be easily extended to **more than two** factors, e.g. $A, B, C$.
- Model then includes higher-order interactions:
  \[
  Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkl}
  \]
  with $\epsilon_{ijkl}$ i.i.d. $N(0, \sigma^2)$ and the usual sum-to-zero constraints.
- Two-way interaction describes how a main-effect depends on the level of the other factor.
- Three-way interaction describes how a two-way interaction depends on level of third factor …
Interpretation

- The higher-order interactions are quite **difficult to interpret**.
- Parameter estimation as before.
- Degrees of freedom of an interaction is the **product of the df’s of the involved factors** (as usual).
- E.g., for the three-way interaction:

\[(a - 1) \cdot (b - 1) \cdot (c - 1)\]